PHYSICAL REVIEW E VOLUME 61, NUMBER 2 FEBRUARY 2000

Parameters characterizing electromagnetic wave polarization

T. Carozzi,* R. Karlsson,† and J. Bergman‡
Swedish Institute of Space Physics, Uppsala Division, SE-755 91 Uppsala, Sweden
(Received 23 December 1998; revised manuscript received 16 September 1999)

In this paper, generalizations of the Stokes parameters and alternative characterizations of three-dimensional (3D) time-varying electromagnetic fields is introduced. One of these characteristics is the normal of the polarization plane, which, in many cases of interest, is parallel (or antiparallel) to the direction of propagation. Others are the two spectral density Stokes parameters which describe spectral intensity and circular polarization. The analysis is based on the spectral density tensor. This tensor is expanded in a base composed of the generators of the SU(3) symmetry group, as given by Gell-Mann and Y. Ne'eman [*The Eight-fold Way* (Benjamin, New York, 1964)] and the coefficients of this expansion are identified as generalized spectral density polarization parameters. The generators have the advantage that they obey the same algebra as the Pauli spin matrices, which is the base for expanding the 2D spectral density tensor with the Stokes parameters as coefficients. The polarization parameters introduced are formulated in the frequency domain, thereby further generalizing the theory to allow for wide-band electromagnetic waves in contrast to the traditional quasi-monochromatic formulation.

PACS number(s): 42.25.Ja, 02.20.Qs

I. INTRODUCTION

The standard description of wave polarization in the transverse plane of propagation of an electromagnetic wave is given by the Stokes parameters [1]. As is well known, these parameters can be found from the two-dimensional coherency tensor, constructed from the transverse components of the wave field. This two-dimensional coherency tensor has some interesting properties: in 1930, Wiener used the unit matrix and the three Pauli spin matrices as base to expand the coherency tensor [2]. Fano later showed that the coefficients in this expansion are the Stokes parameters [3]. The description in terms of the Stokes parameters is straightforward when the direction of arrival is known and the procedure of obtaining them is described by Born and Wolf [4].

If the direction of arrival from the source is unknown *a priori*, we must consider all three field components. In this case, the two-dimensional coherency tensor cannot be used and the Stokes parameters cannot be found directly. Instead, one can use the three-dimensional coherency tensor, or the corresponding tensor in the frequency domain, the spectral tensor, to obtain a complete wave characterization.

There exist several different techniques for finding the degree of polarization, and the axes and the surface normal of the polarization ellipse for n-dimensional fields with $n>2$; see Refs. [5–14]. Means [5] obtains the surface normal of the polarization ellipse directly from the antisymmetric part of the spectral tensor, and does not introduce the Stokes parameters. Roman [6] and Samson [7] utilizes generalizations of the Stokes parameters to higher dimensions than 2. Roman generalizes to three dimensions (3D), by expanding the coherency tensor in terms of nine Hermitean matrices that constitute a Kemmer algebra. Samson [7] generalizes the Stokes parameters to an arbitrary dimension n, where the spectral tensor is expanded in a base of n^2 trace orthogonal, Hermitean tensors. In three dimensions $(n=3)$, the tensors in the expansion represent one set of generators of the special unitary symmetry group SU(3).

The way of characterizing wave polarization presented in the present paper essentially combines the methods [5–7], but instead of concentrating on the spectral tensor and use trace-orthogonal Hermitean tensors or Hermitean tensors obeying Kemmer algebra, we use the concept spectral density tensor and the Hermitean SU(3) generators given by Gell-Mann [15]. Because the SU(3) generators obey the same algebra as the Pauli spin matrices, the expansion of the 2D spectral density matrix in terms of the Pauli spin matrices can be extracted from the 3D spectral density tensor as a limiting case. From the 3D spectral density matrix we can obtain the normal of the plane of polarization and the two spectral density Stokes parameters which describe spectral intensity and circular polarization.

The parameters introduced in this paper provide a powerful description of arbitrary fields for theoretical work but they are also useful in instrumentation where coherent detection of all three components of the electric or magnetic field is possible and where the measured field is completely arbitrary. Useful areas of application could be, e.g., space based radio frequency instruments or characterizing uncollimated light beams.

II. DESCRIPTION OF 3D WAVE POLARIZATION

We are interested in characterizing the sense of polarization of a wave field, and also to obtain the normal of the polarization plane. In order to do so we introduce a set of new 3D polarization parameters and show how they are connected to the usual 2D Stokes parameters and also comprise the normal of the polarization plane.

*Electronic address: tc@irfu.se
†Electronic address: rk@irfu.se
‡Electronic address: jb@irfu.se

A. Spectral density matrix of a vector field

Consider an arbitray electric or magnetic field $\mathbf{f}(\mathbf{r},t)$. We wish to investigate the wave polarization properties of this field at a spatial point. By Fourier transforming the field in the time domain, we decompose the field into its spectral components, which depend on the wave polarization. We define the Fourier transform of the field as

$$\mathbf{F}(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\mathbf{f}(\mathbf{r},t)e^{i\omega t}\,dt \tag{1}$$

and represent this field in terms of three real amplitudes and three real phases, according to

$$\mathbf{F}(\mathbf{r},\omega)=\begin{pmatrix} F_x(\mathbf{r},\omega)\\ F_y(\mathbf{r},\omega)\\ F_z(\mathbf{r},\omega)\end{pmatrix}=\begin{pmatrix} F_1(\mathbf{r},\omega)e^{i\delta_1(\mathbf{r},\omega)}\\ F_2(\mathbf{r},\omega)e^{i\delta_2(\mathbf{r},\omega)}\\ F_3(\mathbf{r},\omega)e^{i\delta_3(\mathbf{r},\omega)}\end{pmatrix}. \tag{2}$$

If, from the outset, the field components are not orthogonal, orthogonalization is performed so that Eq. (2) describe three orthogonal (x,y,z) components of the vector field.

The polarization properties of the wave field can be studied from a second rank tensor formed from the wave field. We will use the *spectral density tensor*, defined as

$$\mathsf{S}_d(\mathbf{r},\omega)=\mathbf{F}\mathbf{F}^{\dagger}=\begin{pmatrix} F_xF_x^* & F_xF_y^* & F_xF_z^*\\ F_yF_x^* & F_yF_y^* & F_yF_z^*\\ F_zF_x^* & F_zF_y^* & F_zF_z^*\end{pmatrix}, \tag{3}$$

where † symbolizes Hermitean conjugate.

Usually the coherency tensor is used to analyze the polarization properties of fields in the time domain, while in the frequency domain, the corresponding tensor used is the spectral tensor. Usage of these tensors requires that the field in question is quasimonochromatic, i.e., are almost monochromatic with a limited bandwidth. On the other hand, when the spectral density tensor is used, there are no limitations on the field. The spectral tensor is formed from the spectral density tensor by applying the operator

$$\hat{B}=\int_{\bar{\omega}-\Delta\omega/2}^{\bar{\omega}+\Delta\omega/2}d\omega \tag{4}$$

that integrates over a small bandwidth $\Delta\omega$, centered around the angular frequency $\bar{\omega}$. We find the spectral tensor, $\mathsf{S}(\mathbf{r},\bar{\omega},\Delta\omega)$, to be given by

$$\mathsf{S}(\mathbf{r},\bar{\omega},\Delta\omega)=\hat{B}\mathsf{S}_d(\mathbf{r},\omega)=\int_{\bar{\omega}-\Delta\omega/2}^{\bar{\omega}+\Delta\omega/2}\mathsf{S}_d(\mathbf{r},\omega)d\omega. \tag{5}$$

The components of the spectral tensor have the physical dimension power, while the spectral density tensor has units power over frequency. One sees that the spectral density tensor is more fundamental than the spectral tensor since no frequency band needs to be specified.

B. Spectral Stokes parameters

By choosing a coordinate system with the z axis along the direction of wave propagation, a transverse wave field can be represented as

$$\mathbf{F}'=\begin{pmatrix} F_x'\\ F_y'\\ 0\end{pmatrix}=\begin{pmatrix} F_1'e^{i\delta_1'}\\ F_2'e^{i\delta_2'}\\ 0\end{pmatrix}. \tag{6}$$

The spectral density tensor obtained from the field in Eq. (6), by omitting the zeros in the third row and the third column and introducing the phase difference $\delta'=\delta_2'-\delta_1'$, takes the form

$$\mathsf{S}_d'(\mathbf{r},\omega)=\begin{pmatrix} (F_1')^2 & F_1'F_2'e^{-i\delta'}\\ F_1'F_2'e^{i\delta'} & (F_2')^2\end{pmatrix}. \tag{7}$$

From Eq. (7), we can introduce the *spectral density Stokes parameters*

$$\mathcal{I}(\mathbf{r},\omega)=(F_1')^2+(F_2')^2, \tag{8a}$$

$$\mathcal{Q}(\mathbf{r},\omega)=(F_1')^2-(F_2')^2, \tag{8b}$$

$$\mathcal{U}(\mathbf{r},\omega)=2F_1'F_2'\cos\delta', \tag{8c}$$

$$\mathcal{V}(\mathbf{r},\omega)=2F_1'F_2'\sin\delta'. \tag{8d}$$

The usual Stokes parameters I, Q, U, and V, see Born and Wolf [4] for the definition, can be found from the spectral density Stokes parameters in Eq. (8) by applying the operator defined in Eq. (4). For example, I is equal to $\hat{B}\mathcal{I}$.

We immediately see that the spectral density tensor can be expressed in terms of the spectral density Stokes parameters as

$$\mathsf{S}_d'=\frac{1}{2}\begin{pmatrix} \mathcal{I}+\mathcal{Q} & \mathcal{U}-i\mathcal{V}\\ \mathcal{U}+i\mathcal{V} & \mathcal{I}-\mathcal{Q}\end{pmatrix}. \tag{9}$$

Note that the total spectral intensity $\mathcal{I}$ is equal to the trace of the spectral density tensor, $\mathcal{I}=\mathrm{Tr}(\mathsf{S}_d')$. We also know that the tensor in Eq. (9) can be written as a linear combination of the three generators of the special unitary symmetry group SU(2), i.e., the three Pauli spin matrices σ_i, and the unit matrices 1_2 [2], with the spectral density Stokes parameters as scalar coefficients [3]:

$$\mathsf{S}_d'=\frac{1}{2}(\mathcal{I}1_2+\mathcal{U}\sigma_1+\mathcal{V}\sigma_2+\mathcal{Q}\sigma_3). \tag{10}$$

The determinant of the spectral density tensor in Eq. (7) is equal to zero, and so also the determinant of Eq. (9). The following relation is thus obtained:

$$\mathcal{Q}^2+\mathcal{U}^2+\mathcal{V}^2=\mathcal{I}^2. \tag{11}$$

For the usual Stokes parameters, we instead of Eq. (11) have the relation $Q^2+U^2+V^2\leqslant I^2$ [4], where the equality only hold for a monochromatic field.

C. Generalized polarization parameters

To generalize Eq. (9) to three dimensions, we use the generators of the SU(3) symmetry group to form a new representation of the spectral density tensor. The unit matrix in three dimensions, 1_3 and the generators, $\hat{\lambda}_i$, $i=1,\ldots,8$, given by Gell-Mann [15], will be used. The spectral density tensor is formed as a linear combination of the unit matrix and the generators, and for the scalar coefficients we use the symbols Λ_i, $i=0,\ldots,8$:

$$\mathsf{S}_d=\frac{1}{3}\Lambda_0 1_3+\frac{1}{2}\sum_{i=1}^{8}\Lambda_i\hat{\lambda}_i=\begin{pmatrix} \frac{1}{3}\Lambda_0+\frac{1}{2}\Lambda_3+\frac{1}{2\sqrt{3}}\Lambda_8 & \frac{1}{2}\Lambda_1-i\frac{1}{2}\Lambda_2 & \frac{1}{2}\Lambda_4-i\frac{1}{2}\Lambda_5 \\ \frac{1}{2}\Lambda_1+i\frac{1}{2}\Lambda_2 & \frac{1}{3}\Lambda_0-\frac{1}{2}\Lambda_3+\frac{1}{2\sqrt{3}}\Lambda_8 & \frac{1}{2}\Lambda_6-i\frac{1}{2}\Lambda_7 \\ \frac{1}{2}\Lambda_4+i\frac{1}{2}\Lambda_5 & \frac{1}{2}\Lambda_6+i\frac{1}{2}\Lambda_7 & \frac{1}{3}\Lambda_0-\frac{1}{\sqrt{3}}\Lambda_8 \end{pmatrix}. \qquad (12)$$

We call the coefficients Λ_i *generalized spectral density polarization parameters*. It can be seen that the trace of Eq. (12) is equal to Λ_0 and we identify the spectral intensity $\mathcal{I}=\Lambda_0$. The normalization in Eq. (12) is such that it reduces to the spectral Stokes parameters for the case of a plane wave chosen to propagate along the z axis. In this case $\Lambda_4=\Lambda_5=\Lambda_6=\Lambda_7=0$ and $\Lambda_8=(1/\sqrt{3})\Lambda_0$. Inserting these expressions into Eq. (12), we obtain

$$\mathsf{S}_d=\frac{1}{2}\begin{pmatrix} \Lambda_0+\Lambda_3 & \Lambda_1-i\Lambda_2 & 0 \\ \Lambda_1+i\Lambda_2 & \Lambda_0-\Lambda_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (13)$$

which is the same as Eq. (9), except for the name of the parameters.

III. PHYSICAL INTERPRETATION OF THE PARAMETERS

The spectral density tensor, which was introduced previously, can be seen to consist of several parts which can be ascribed specific physical meaning that we now will discuss.

A. The normal of the polarization plane

To the antisymmetric part of a tensor, a dual pseudovector is associated, see Arfken and Weber [16]. The pseudovector of the spectral density tensor in Eq. (12) is given by $i(-\Lambda_7,\Lambda_5,-\Lambda_2)$. We introduce a similar vector $\mathbf{V}$

$$\mathbf{V}\equiv(\Lambda_7,-\Lambda_5,\Lambda_2), \qquad (14)$$

which by definition is real. Comparing the expressions for the spectral density tensors in Eqs. (3) and (12), we obtain

$$\Lambda_7=-2\,\mathrm{Im}\{F_yF_z^*\}, \qquad (15a)$$

$$\Lambda_5=-2\,\mathrm{Im}\{F_xF_z^*\}, \qquad (15b)$$

$$\Lambda_2=-2\,\mathrm{Im}\{F_xF_y^*\}. \qquad (15c)$$

The time-dependent field vector $\mathbf{f}(\mathbf{r},t)$ traces the polarization ellipse. The complex vector $\mathbf{F}(\mathbf{r},\omega)$ and its complex conjugate $\mathbf{F}^*(\mathbf{r},\omega)$ form a polarization plane in space. This plane is the same plane as the polarization ellipse defines. Because

$$\mathbf{V}\cdot\mathbf{F}=\mathbf{V}\cdot\mathbf{F}^*=0, \qquad (16)$$

$\mathbf{V}$ is perpendicular to $\mathbf{F}$, and thereby also perpendicular to the polarization plane. It can be shown (see Lindell [17]) that a vector normal to the plane of polarization is parallel to $i\mathbf{F}\times\mathbf{F}^*$, and the magnitude of this vector is $2/\pi$ times the area of the polarization ellipse. Using Eqs. (2), (3), and (12), we obtain

$$i\mathbf{F}\times\mathbf{F}^*=(\Lambda_7,-\Lambda_5,\Lambda_2)=\mathbf{V}, \qquad (17)$$

i.e., $|\mathbf{V}|$ is equal to $2/\pi$ times the area of the polarization ellipse.

The normal of the polarization plane gives, in the case of transverse waves, the direction of wave propagation. Defining right- and left-hand polarization as the polarization seen by the wave itself and not by an observer looking at the approaching wave, the normal of the polarization plane $\mathbf{V}$ is parallel to the direction of propagation for a right-hand polarized wave and antiparallel for a left-hand polarized. The orientation of the plane of polarization can be specified with two angles. Referring to Fig. 1, the first angle is the angle α between the plane of polarization and the xy plane. The second is the angle β between the intersection of the planes and the x axis.

B. Number of independent parameters

In general, six parameters are needed in order to characterize a wave field, for example three complex numbers or three real amplitudes and three real phases. The four 2D spectral Stokes parameters $\mathcal{I}$, $\mathcal{Q}$, $\mathcal{U}$, and $\mathcal{V}$, characterize the polarization in the transverse plane of propagation. The propagation direction adds another two parameters to the spectral Stokes parameters, giving a total number of six parameters.

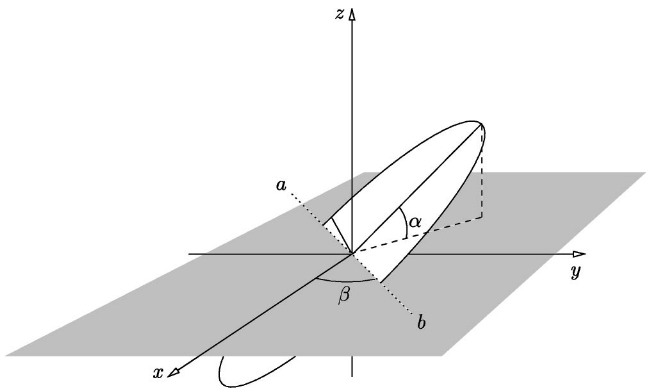

FIG. 1. Orientation of the polarization plane. The line ab is the intersection between the plane of polarization and the xy plane, and the angle between these planes is α. The angle β specifies where in the xy plane the two planes intersect.

When the spectral density tensor is formed, the overall phase is lost and the field can be multiplied with an arbitrary phase without changing the spectral density tensor. The number of independent parameters can therefore only be five and for the spectral density Stokes parameters the loss of an independent parameter is described by Eq. (11). This also means that only five of the nine generalized spectral density polarization parameters can be independent.

By considering the scalar invariants of the spectral density tensor, we can verify that there is five independent parameters. The scalar coefficients in the secular equation, $\det(\mathsf{S}_d - \lambda \mathsf{1}_3) = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$, of a tensor S_d are independent of the base vectors, and are called scalar invariants. The scalar invariants of a tensor are the trace (I_1), the sum of the three cofactors (I_2), and the determinant (I_3). The 2D and 3D spectral density tensors use different base vectors for the wave field, but are equivalent and must have the same invariants. Further, the symmetric and the antisymmetric parts of the spectral density tensor are never mixed and in turn they must have their own invariants.

The first invariant, the trace, gives just the spectral intensity. Calculating the second invariant gives two equations, one for the symmetric part and one for the antisymmetric. The third invariant adds another two equations, resulting in a number of four equations that reduce the number of independent generalized spectral density polarization parameters from 9 to 5. One of these equations is specially interesting, since it gives the magnitude of the spectral density Stokes parameter $\mathcal{V}$:

$$|\mathcal{V}| = \sqrt{\Lambda_7^2 + \Lambda_5^2 + \Lambda_2^2} = |\mathbf{V}|. \tag{18}$$

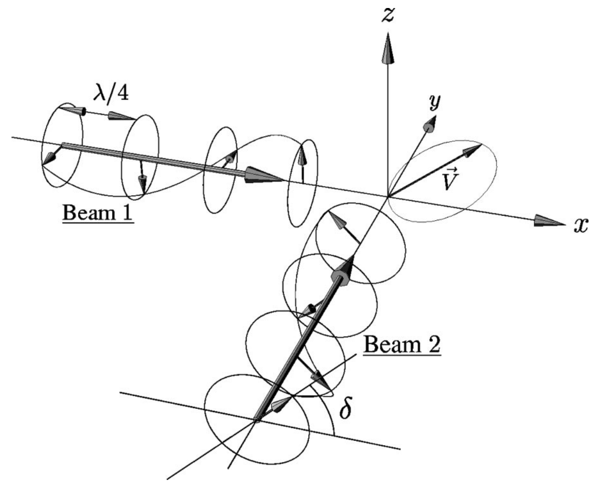

FIG. 2. Two right-hand circularly polarized beams intersecting each other at right angles. In this example, beam 2 has a phase shift δ with respect to beam 1. As a function of the phase shift, δ, the $\mathbf{V}$ vector traces out an ellipse depicted at the origin. The figure shows the case when $\delta = \pi/4$ and $\mathbf{V} = (1 + 1/\sqrt{2}, 1 + 1/\sqrt{2}, 1/\sqrt{2})$.

This tells us that Λ_7, Λ_5, Λ_2, and $|\mathbf{V}|$ all describe circular polarization, and that $|\mathbf{V}|$ is an invariant.

IV. EXAMPLE

We now present a simple example on how the polarization parameters discussed in this article can be used to characterize an electric field: consider two circularly polarized monochromatic beams of unit amplitude which intersect one another at right angles, see Fig. 2. As a simplification, we model the beams as plane waves and allow only one degree of freedom, namely, the relative phase δ, between the two plane waves. Let one of the wave vectors lay along the x axis and the other along the y axis, intersecting each other at the origin. Assume right-hand circularly polarized beams, with fields represented by

$$\mathbf{E}_1^T = \begin{pmatrix} 0 & -1 & -i \end{pmatrix} \quad \text{and} \quad \mathbf{E}_2^T = e^{i\delta}\begin{pmatrix} 1 & 0 & -i \end{pmatrix}. \tag{19}$$

At the intersection point the total electric field can be written as

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \tag{20}$$

and the resulting spectral density tensor is

$$\begin{pmatrix} 1 & -\cos\delta - i\sin\delta & -\sin\delta + i(1+\cos\delta) \\ -\cos\delta + i\sin\delta & 1 & -\sin\delta - i(1+\cos\delta) \\ -\sin\delta - i(1+\cos\delta) & -\sin\delta + i(1+\cos\delta) & 2(1+\cos\delta) \end{pmatrix}. \tag{21}$$

We can easily extract

$$\mathbf{V}=\begin{pmatrix} 1+\cos\delta \\ 1+\cos\delta \\ \sin\delta \end{pmatrix}. \tag{22}$$

Figure 2 shows $\mathbf{V}$ for the case $\delta=\pi/4$.

V. CONCLUSION

In this paper we have introduced generalizations of the Stokes parameters and alternative descriptions of three dimensions time-varying electromagnetic fields. It has been shown how, given a vector field in a Cartesian base, it is possible to determine parameters which characterize the polarization of a wave in a simple, yet meaningful way.

By expanding the spectral density tensor in a base consisting of the SU(3) generators given by Gell-Mann [15], a generalization of the Stokes parameters to three-dimensions is obtained. The 2D spectral density tensor, comprising the Stokes parameters, is obtained as a limiting case when the direction of arrival is known. Also, the normal vector of the polarization plane, $\mathbf{V}$, which gives the direction of arrival for a transverse wave, is obtained directly from the 3D spectral density tensor together with the spectral density Stokes parameters $\mathcal{V}$ and $\mathcal{I}$. The orientation of the polarization plane is given by the angles α and β shown in Fig. 1.

These alternative descriptions have the advantage that it is not necessary to specify a ''viewing direction'' and that the fields from two or more sources in different directions can be measured simultaneously. Also, some of the parameters introduced are invariants of the theory and do not depend on the particular choice of coordinate system. Finally, the spectral density tensor is used and no information is lost by integrating over an angular frequency interval to obtain the spectral tensor.

ACKNOWLEDGMENTS

We gratefully acknowledge the assistance given by Bo Thidé and also the financial support from the Swedish Natural Science Research Council (NFR). One of the authors (R.K.) gratefully acknowledges financial support from the Advanced Instrumentation and Measurements (AIM) graduate school.

[1] G. G. Stokes, Trans. Cambridge Philos. Soc. **9**, 399 (1852); *Mathematical and Physical Papers* (Cambridge University Press, Cambridge, England, 1901), Vol. III, p. 233.
[2] N. Wiener, Acta Math. **55**, 117 (1930).
[3] U. Fano, Phys. Rev. **93**, 121 (1954).
[4] M. Born and E. Wolf, *Principles of Optics*, 6th (corr) ed. (Cambridge University Press, Cambridge, England, 1998).
[5] J. D. Means, J. Geophys. Res. **77**, 5551 (1972).
[6] P. Roman, Nuovo Cimento **13**, 2546 (1959).
[7] J. C. Samson, Geophys. J. R. Astron. Soc. **34**, 403 (1973).
[8] J. C. Samson, Geophys. J. R. Astron. Soc. **51**, 583 (1977).
[9] J. C. Samson, J. Geophys. Res. **48**, 195 (1980).
[10] J. C. Samson and J. V. Olson, Geophys. J. R. Astron. Soc. **61**, 115 (1980).
[11] J. C. Samson and J. V. Olson, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **40**, 137 (1981).
[12] J. C. Samson and J. V. Olson, Geophys. J. **46**, 1423 (1981).
[13] K. Kodera, R. Gendrin, and C. de Villedary, J. Geophys. Res. **82**, 1245 (1977).
[14] H. P. Ladreiter *et al.*, Radio Sci. **30**, 1699 (1995).
[15] M. Gell-Mann and Y. Ne'eman, *The Eight-fold Way* (Benjamin, New York, 1964).
[16] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists* (Academic Press, Boston, 1995).
[17] I. V. Lindell, Int. J. Electr. Eng. Educ. **20**, 33 (1983).